Math 1552

Review of integration techniques

Math 1552 lecture slides adapted from the course materials
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Review of integration methods (so far)

What techniques have we seen so far to evaluate definite and indefinite integrals?

- Direct integration (know/memorize common formulas) →
- Substitution, or u-subs
- Integration by parts, or IBP
- Powers and products of trig functions
- Trigonometric substitutions, or trig subs
- Partial fractions

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \sin(x)\,dx, \int \cos(x)\,dx$$

trig identities:

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

Other topics we covered:

Riemann sums, FTC, areas between curves, L'Hopital's rule, and improper integrals

Hints and suggestions

- Practice picking out the relevant methods of integration on the review sheet (***bring to class ~~next~~ lecture for Q/A☺***)
- When in doubt, try using a u-sub first to simplify the integral
- You may need to combine multiple techniques we have seen, for example, a u-sub followed by IBP and then a term that involves partial fractions (see ***Example R.3*** in the next slides)
- Review key components of each method to study

Method of substitution (u-subs)

This method is the reverse of the chain rule for derivatives:

$F'(x) = f(x)$

Let F be an antiderivative of f. Let $u = g(x)$.

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du = F(u) + C = F(g(x)) + C$$

$u = g(x)$, $du = g'(x) dx$

In other words:

$$\int f(stuff) \cdot (stuff)' dx = F(stuff) + C$$

Substitution with Definite Integrals

To evaluate $\int_a^b f(g(x))g'(x)dx$,

set $u = g(x)$ and *change the limits of integration* to match the new variable:

$$\int_a^b \underbrace{f(g(x))}_{f(u)}\underbrace{g'(x)dx}_{du} = \underbrace{\int_{g(a)}^{g(b)} f(u)du}$$

in The ex. (R.2)
$u = \sin(2t)$
with $a = -\frac{\pi}{8}$,
$b = \frac{\pi}{2}$
$\rightarrow g(t) = \sin(2t)$

Example

Evaluate the following integral: $\int \underline{x^2} \sin(x^3 + 5)\underline{dx} = I$

$$\underbrace{x^2 \, dx}_{\frac{du}{3}}$$

Which integration method to invoke? (Explain)

u-sub first: $u = x^3 + 5$

$du = 3x^2 dx$

$$\rightarrow I = \frac{1}{3} \cdot \int \sin(u) du$$

$$= -\frac{1}{3} \cdot \cos(u) + C$$

$$= -\frac{1}{3} \cos(x^3 + 5) + C$$

Example R.1

Evaluate the following integral: $\int x\sqrt{x+10}\,dx = I$

Which integration method to invoke? (Explain)

ideas (ways from lecture):

(1) perform a u-sub ✓

(2) OR IBP taking $u = x$

two choices!

(1) $u = x$ ✗

(2) $u = x+10$, $du = dx$

$\to$ use a u-sub: choose $u = x+10 \leftrightarrow x = u-10$

$du = dx$

$$I = \int (u-10)u^{1/2}\,du = \int u^{3/2}\,du - 10\int u^{1/2}\,du$$

$$= \frac{2}{5}u^{5/2} - \frac{10\cdot 2}{3}u^{3/2} + C$$

$$\frac{u^{5/2}}{3/2+1}$$

$$= \frac{2}{5}(x+10)^{5/2} - \frac{20}{3}(x+10)^{3/2} + C$$

When to Use Partial Fractions:

$\int \frac{P_1(x)}{P_2(x)} dx$ where P_1, P_2 are Polynomials

Use the method of partial fractions to evaluate the integral of a *rational function* when:

- The degree of the numerator is *less than* that of the denominator.
- The denominator can be *completely factored* into linear and/or irreducible quadratics

Partial Fractions Procedure:

$P_2(x) = 2x^2 + 4x + 2$

$= 2(x^2 + 2x + 1)$

1. If the leading coefficient of the denominator is not a "1", factor it out.
2. If the degree of the numerator is greater than that of the denominator, carry out long division first. (see lecture notes for an example)
3. Factor the denominator completely into linear and/or irreducible quadratic terms.

Partial Fractions Procedure:

4. For each linear term of the form $(x-a)^k$, you will have *k* partial fractions of the form:

$$\boxed{\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_k}{(x-a)^k}}$$

(Note: if *k*=1, there is only one fraction to handle, etc.)

$$\frac{1}{(x-1)^3} \longrightarrow \frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{A_3}{(x-1)^3} \quad (k=3 \text{ here})$$

Partial Fractions Procedure:

5. For each irreducible quadratic term of the form $(x^2+bx+c)^m$, you will have m partial fractions of the form:

$$\boxed{\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{\left(x^2+bx+c\right)^2}+\frac{A_3x+B_3}{\left(x^2+bx+c\right)^3}+\ldots+\frac{A_mx+B_m}{\left(x^2+bx+c\right)^m}}$$

(Note: if m=1, there is only one fraction, etc.)

$$\frac{1}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2} \qquad (m=2)$$

Partial Fractions Procedure:

6. Solve for all the constants A_i and B_i. To solve:
   - Multiply everything by the common denominator.
   - Combine all like terms, then solve equations for all the A_i and B_i terms; OR plug in values to find equations for A_i and B_i terms.
7. Integrate using all the integration methods we have learned.

Example R.2

Evaluate the following integral: $\displaystyle\int_{-\frac{\pi}{8}}^{\cancel{\frac{\pi}{4}}\ \pi/2} \frac{\cos(2t)}{\sin^2(2t) + 3\sin(2t) - 4}\,dt = I$

Which integration method to invoke? (Explain)

$\rightarrow$ use a u-sub first:

$$u = \sin(2t),\ du = 2\cos(2t)\,dt$$

$$\sin\left(2\cdot\frac{\pi}{2}\right) = \sin(\pi) = 0$$

$$\sin(-\pi/4) = -\frac{\sqrt{2}}{2}$$

$$\rightarrow I = \frac{1}{2}\int_{-\frac{\sqrt{2}}{2}}^{0} \frac{du}{u^2+3u-4}$$

$n^2+3n-4=(n-1)(n+4)$

$\rightarrow$ apply partial fractions:

$$\frac{1}{(n-1)(n+4)}=\frac{A}{n-1}+\frac{B}{n+4}$$

$$1=A(n+4)+B(n-1) \longrightarrow \text{choose } n=+1,-4$$

<u>when $n=+1$:</u> $1=5A \longleftrightarrow A=\frac{1}{5}$

<u>when $n=-4$:</u> $1=-5B \longleftrightarrow B=-1/5$

$$\frac{A}{n-1}+\frac{B}{n+4}=\frac{1}{5}\left[\frac{1}{n-1}-\frac{1}{n+4}\right] \uparrow$$

$$I = \frac{1}{10}\int_{-\frac{\sqrt{2}}{2}}^{0}\left[\frac{1}{x-1} - \frac{1}{x+4}\right]dx$$

$\int \frac{dx}{x} = \ln|x| + C$

$$= \frac{1}{10}\Big[\ln|x-1| - \ln|x+4|\Big]\Bigg|_{-\frac{\sqrt{2}}{2}}^{0}$$

$$= \frac{1}{10}\ln\left|\frac{x-1}{x+4}\right|\Bigg|_{-\frac{\sqrt{2}}{2}}^{0}$$

$\rightarrow$ apply the FTC

$$= \frac{1}{10}\left[\ln\left|\frac{1}{4}\right| - \ln\left|\frac{1+\frac{\sqrt{2}}{2}}{4-\frac{\sqrt{2}}{2}}\right|\right]$$

(leave the answer in this form)

if we hadn't changed the bounds of int!

$$I = \frac{1}{10}\ln\left|\frac{\sin(2t)-1}{\sin(2t)+4}\right|\Bigg|_{-\pi/8}^{\pi/2} = \text{same answer}$$

Integration by Parts - Summary

- To apply (*), need to identify u, dv, + need to be able to find v from dv.
- use ILATE rule to select the function u

Integration by parts comes from the product rule for differentiation.

$$\boxed{\int u \cdot dv = uv - \int v \cdot du} \quad (*)$$

Differentiate u to obtain du.

Find v by taking an antiderivative of dv.

$$(fg)' = f'g + fg' \implies$$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Example R.3

Evaluate the following integral: $I = \int_0^1 \ln\left(1 + \overset{u}{x^{1/4}} + \overset{u^2}{x^{1/2}}\right) \boxed{dx} \cdot \frac{4x^{3/4}}{\boxed{4x^{3/4}}} \leftarrow 4u^3$ $\;du$

Which integration method to invoke? (Explain)

$\rightarrow$ use a u-sub first:

$u = x^{1/4}$, $du = \frac{1}{4 \cdot x^{3/4}} dx$

$u(0) = 0$, $u(1) = 1$

$\int \frac{dx}{1+e^x}$, $u = e^x$, $du = e^x dx$

$= \int \frac{du}{u(u+1)}$

$\rightarrow I = 4 \int_0^1 u^3 \ln(1+u+u^2)\, du$ $(s = u)$

$= 4 \int_0^1 s^3 \cdot \ln(1+s+s^2)\, ds$

(hint: apply IBP to the last line)

ILATE

$\uparrow \ \ \ \llcorner s^3$

$\ln(1+s+s^2)$

$$u = \ln(1+s+s^2) \qquad dv = s^3\,ds$$

$$du = \frac{2s+1}{1+s+s^2}\,ds \qquad v = \frac{s^4}{4}$$

$$\left(\underline{\text{IBP}}: \int u\,dv = uv - \int v\,du\right)$$

$$I = \underbrace{4 \cdot \frac{s^4}{4} \cdot \ln(1+s+s^2)\Big|_{s=0}^{s=1}}_{1 \cdot \ln(3) - 0 \cdot \ln(1) = \ln(3)} - \underbrace{\frac{4}{4}\int_0^1 \frac{s^4(2s+1)\,ds}{1+s+s^2}}_{I_2}$$

$$I_2 = \int_0^1 \frac{s^4(2s+1)\,ds}{1+s+s^2}$$

$\longrightarrow$ apply PF, long division first

$$\begin{array}{r}
2s^3 - s^2 \quad -s \quad +2 \\
1+s+s^2 \;\big)\; 2s^5 + s^4 + 0\cdot s^3 + 0\cdot s^2 + 0s + 0 \\
-\left(2s^5 + 2s^4 + 2s^3\right) \\
\hline
-s^4 - 2s^3 + 0\cdot s^2 \\
-\left(-s^4 - s^3 - s^2\right) \\
\hline
\end{array}$$

$$-s^3 + s^2 + 0 \cdot s$$

$$\underline{-(-s^3 - s^2 - s)}$$

$$2s^2 + s \quad + 0$$

$$\underline{-(2s^2 + 2s + 2)}$$

$$-s - 2$$

In total: $s^4(2s+1) = (1+s+s^2)(2s^3 - s^2 - s + 2)$

$$-(s+2)$$

$$I_2 = \underbrace{\int_0^1 (2s^3 - s^2 - s + 2)\,ds}_{\to \text{ power rule, then FTC}} - \int_0^1 \frac{(s+2)}{1+s+s^2}\,ds$$

complete the square (2) for I_3

$$(2):\ 1+s+s^2 = (s+b)^2 + c$$
$$= (s+1/2)^2 + 3/4$$

$$\to I_3 = \int_0^1 \frac{(s+2)\,ds}{1+s+s^2}$$

v-sub: $v = s + 1/2$
$dv = ds$

$$= \int_{1/2}^{3/2} \frac{(v + 3/2)\,dv}{v^2 + 3/4}$$

$$= \underbrace{\int_{1/2}^{3/2} \frac{v\,dv}{v^2+3/4}}_{\text{eval. by } q\text{-sub: } q = v^2+3/4\ (\cdots)} + \underbrace{\frac{3}{2}\int_{1/2}^{3/2} \frac{dv}{v^2+3/4}}_{\text{factor out the } 3/4 \text{ from the denom, then eval in terms of } \tan^{-1}}$$

- Out of order Q from the chat

for Riemann sums, know these formulas:

$$\sum_{k=0}^{N} c\cdot k = c\cdot(N+1), \text{ c a constant}$$

$$\sum_{k=0}^{N} k = \frac{1}{2}N(N+1)$$

$$\sum_{k=0}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$ (would be reminded of this formula)

Antiderivatives of powers and products of trig functions

$(*) \sin^2 x + \cos^2 x = 1$

$(*) 1 + \tan^2 x = \sec^2 x$

$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

$(*) \sin^2 x = \frac{1}{2}[1 - \cos(2x)]$

$(*) \cos^2 x = \frac{1}{2}[1 + \cos(2x)]$

$(*) \sin(2x) = 2 \sin x \cos x$

$\sin x \cos y = \frac{1}{2}[\sin(x - y) + \sin(x + y)]$

$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$

good to know, should not need for the midterm

What to expect with powers / products of trig functions:

For integrals of the form

$$\int \cos^n(x)\sin^m(x)dx$$

OR

$$\int \tan^n(x)\sec^m(x)dx$$

we need to apply appropriate trig identities from the last slide to handle respective separate cases of n and m even or odd.

Apply other identities for integrals of the form

$$\int \cos(ax)\cos(bx)dx$$

OR

$$\int \sin(ax)\sin(bx)dx$$

OR

$$\int \cos(ax)\sin(bx)dx$$

Example R.5

Evaluate the following integral: $I = \int_{-\frac{1}{3}}^{\frac{1}{6}} \sin^2(\pi x)\cos^5(\pi x)\,dx$ *(Sketch solution)*

Which integration method to invoke? (Explain)

→ idea: apply a trig. ident. followed by a u-sub. so that the integrand looks like poly(u)·du.

$\rightarrow \cos^5(\pi x) = \cos(\pi x)\left(1-\sin^2(\pi x)\right)^2$

→ Now apply a u-sub:

$u = \sin(\pi x),\quad du = \pi \cdot \cos(\pi x)\,dx$

$$u\left(\frac{1}{6}\right) = \sin\left(\frac{\pi}{6}\right) = 1/2$$

$$u\left(-\frac{1}{3}\right) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\rightarrow I = \frac{1}{\pi}\int_{-\frac{\sqrt{3}}{2}}^{1/2} u^2(1-u^2)^2\,du$$

$$= \frac{1}{\pi}\int_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}} (u^2 - 2u^4 + u^6)\,du$$

$$= \frac{1}{\pi}\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right]\Bigg|_{-\sqrt{3}/2}^{1/2} \rightarrow \text{apply the FTC}$$

Trigonometric Substitutions (trig subs)

We use a trig substitution when no other integration method will work, and when the integral contains one of these types of terms:

$$a^2 - x^2$$

$$x^2 - a^2$$

$$a^2 + x^2$$

Trig subs - Form 1:

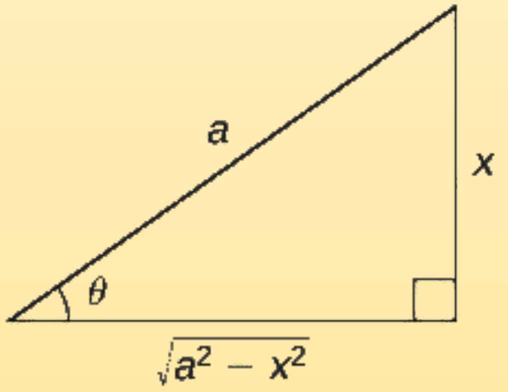
When the integral contains a term of the form

$$a^2 - x^2, \quad \sqrt{a^2-x^2}$$

use the substitution:

$$\boxed{x = a\sin\theta}$$

$$\sin\theta = \frac{x}{a}$$



$$dx = a\cdot\cos\theta\, d\theta$$

$$a^2 - x^2 = a^2(1-\sin^2\theta) = a^2\cdot\cos^2(\theta)$$

Credits for figure: https://math.libretexts.org/Bookshelves/Calculus

Trig subs - Form 2:

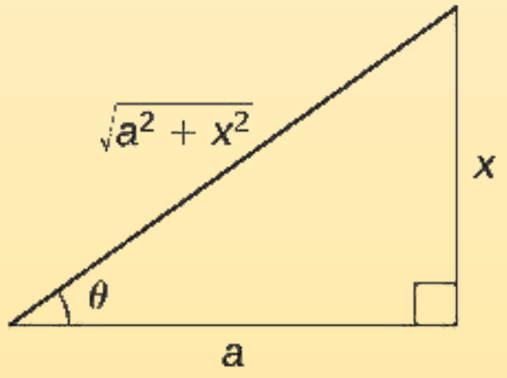
When the integral contains a term of the form

$$a^2 + x^2,$$

use the substitution:

$$\boxed{x = a\tan\theta}$$

$$\tan\theta = \frac{x}{a}$$



$$dx = a \cdot \sec^2\theta \, d\theta$$

$$a^2 + x^2 = a^2(1 + \tan^2\theta)$$

$$= a^2 \cdot \sec^2\theta$$

Credits for figure: https://math.libretexts.org/Bookshelves/Calculus

Trig subs - Form 3:

When the integral contains a term of the form

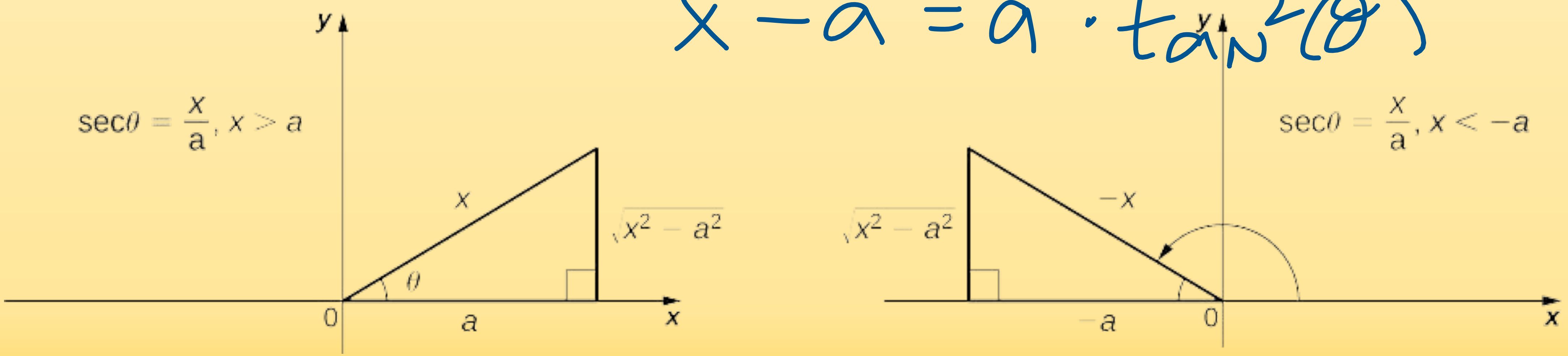
$$x^2 - a^2,$$

use the substitution:

$$\boxed{x = a\sec\theta}$$

$dx = a\tan(\theta)\cdot\sec\theta\, d\theta$

$x^2 - a^2 = a^2\cdot\tan^2(\theta)$



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Example R.6

Evaluate the following integral: $\int \sqrt{25+x^2}\,dx = I$

Which integration method to invoke? (Explain)

$\rightarrow$ apply the method of trig sub $(a=5)$

$x = 5\cdot\tan\theta\,,\ dx = 5\cdot\sec^2\theta\,d\theta\,,$

$\sqrt{x^2+25} = 5\cdot\sec\theta$

$\rightarrow I = 25\underbrace{\int \sec^3\theta\,d\theta}_{\rightarrow \text{IBP}(I_2)}$

To eval I_2 by parts:

$u = \sec(\theta)$ $\qquad dv = \sec^2(\theta)\,d\theta$

$du = \tan(\theta)\sec(\theta)\,d\theta$ $\qquad v = \tan(\theta)$

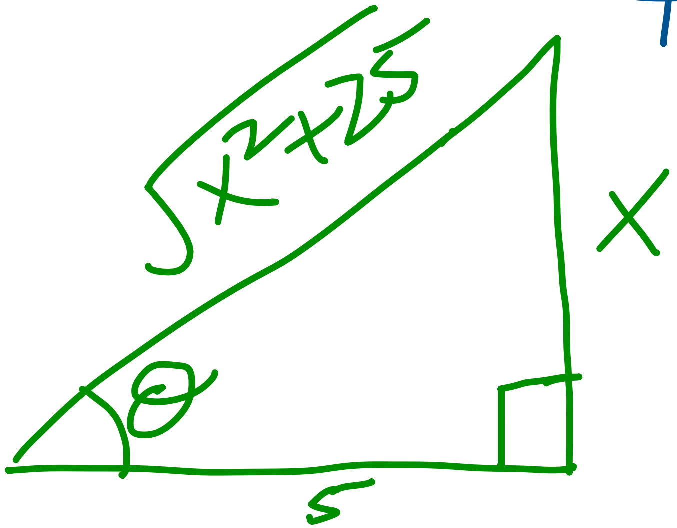
$$I_2 = \sec\theta \cdot \tan\theta - \int \underbrace{\tan^2\theta}_{(\sec^2\theta - 1)} \cdot \underbrace{\sec\theta}_{\sec\theta}\, d\theta$$

$$= \sec\theta \cdot \tan\theta - \underbrace{\int \sec^3\theta\, d\theta}_{I_2} + \int \sec\theta\, d\theta$$

$$I_2 = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\tan\theta + \sec\theta\right| + C$$

$$\text{So } I = \frac{25}{2}\left(\sec\theta\tan\theta + \ln\left|\tan\theta + \sec\theta\right|\right) + C$$

$$\tan\theta = \frac{x}{5}$$



$$, \ \sec\theta = \frac{\sqrt{x^2+25}}{5}$$

$$\underline{I} = \frac{25}{2}\left(\frac{x\sqrt{x^2+25}}{25} + \ln\left|\frac{x}{5} + \frac{\sqrt{x^2+25}}{5}\right|\right) + C$$

Improper integrals

A definite integral is improper if:

- The function has a vertical asymptote at $x=a$, $x=b$, or at some point c in the interval (a,b).
- One or both of the limits of integration are infinite (positive or negative infinity).

Convergence of an Integral

- If an improper integral evaluates to a **finite number**, we say it ***converges***.

- If the integral evaluates to $\pm\infty$ or to, $\infty - \infty$, we say the integral ***diverges***.

Case 1: At Least One Infinite Limit

Redefine the integral into one of the following.

$$(i)\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

$$(ii)\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

$$(iii)\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx = \lim_{\substack{a\to-\infty \\ b\to+\infty}}\int_a^b f(x)dx$$

and now use parts (i) and (ii).

for $c \in [a,b]$

Case 2: $f(c) \to \infty$ Between a and b

- Case 2 occurs when f has a vertical asymptote on the interval [a,b].
- Redefine the integral into one of the following.

$= \pm\infty$

$$(i)\ \text{If } f(a)\ \cancel{\text{DNE}}\text{, then}: \int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$$

$\pm\infty$

$$(ii)\ \text{If } f(b)\ \cancel{\text{DNE}}\text{, then}: \int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx$$

$\pm\infty$

$$(iii)\ \text{If } f(c)\ \cancel{\text{DNE}}\text{, where } a < c < b\text{, then}:$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{b \to c^-} \int_a^b f(x)dx + \lim_{a \to c^+} \int_a^b f(x)dx$$

and now use parts (i) and (ii).

Example R.7

Evaluate the following integral: $\int_0^{\frac{1}{2}} \left[\pi\left(x-\frac{1}{2}\right)\sec^2(\pi x)+\tan(\pi x)\right] dx = I$

Which integration method to invoke? (Explain)

$$I = I_1 + I_2 + I_3$$

$$\underline{\hphantom{I_2}}$$

$$I_2 = -\int_0^{1/2} \frac{\pi}{2} \cdot \sec^2(\pi x)\,dx$$

$$= -\frac{1}{2}\tan(\pi x)\Big|_0^{1/2}$$

$$I_3 = \int_0^{1/2} \tan(\pi x)\,dx = \frac{1}{\pi} \ln\left|\sec(\pi x)\right| \Big|_0^{1/2}$$

$$I_1 = \int_0^{1/2} \pi x \cdot \sec^2(\pi x)\,dx \longrightarrow \text{eval using IBP}$$

$u = \pi x$

$du = \pi\,dx$

$dv = \sec^2(\pi x)\,dx$

$v = \frac{1}{\pi} \tan(\pi x)$

$$= x\tan(\pi x)\Big|_0^{1/2} - \int_0^{1/2} \tan(\pi x)\,dx$$

So $I = \left(x-\frac{1}{2}\right)\tan(\pi x)\Big|_0^{1/2}$

$\to$ problem at $x = \frac{1}{2}$

$\to$ Need the value of the limit:

$$\lim_{x \to \frac{1}{2}^-} \left(x - \frac{1}{2}\right) \tan(\pi x)$$

$$= \lim_{x \to \frac{1}{2}^-} \frac{\left(x - \frac{1}{2}\right) \sin(\pi x)}{\cos(\pi x)}$$

$\frac{0}{0}$, use L'Hopital

$$= \lim_{x \to \frac{1}{2}^-} \left[\frac{\sin(\pi x) + \left(x - \frac{1}{2}\right) \cos(\pi x)}{-\pi \cdot \sin(\pi x)} \right]$$

$$= -\frac{1}{\pi} + 0 = -\frac{1}{\pi}$$

$$I = -\frac{1}{\pi} - \left(0 - \frac{1}{2}\right)\tan(0)$$

$$= -\frac{1}{\pi}$$

Math 1552

Review for the Midterm Exam

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Let's open things up for general questions and specific questions on the review sheet?

(List and enumerate problems) (review of the FTC)

1. if F is an antiderivative of f, then

$$\int f(x)\,dx = F(x) + C$$

2. if F is an antiderivative f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$